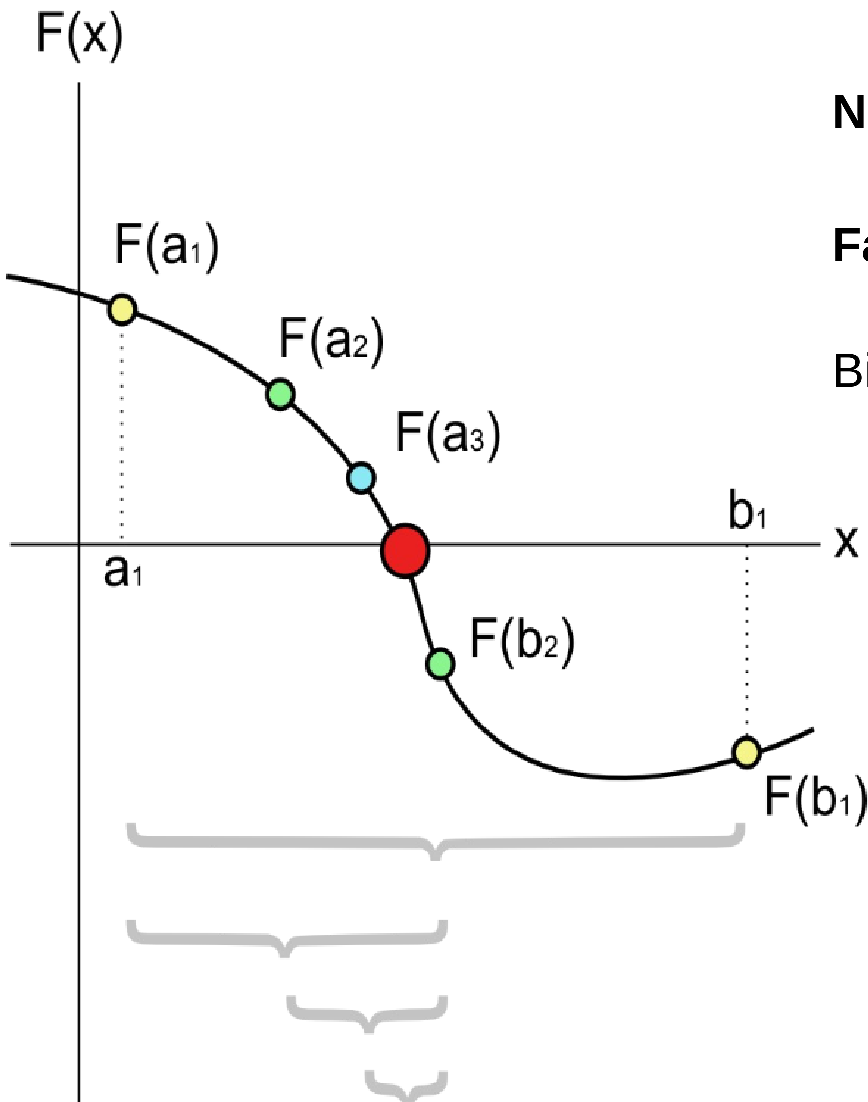
# NUMERICAL METHODS

# Roots of Transcendental equations

- An equation or formula involving transcendental functions.
- Examples of transcendental functions include the exponential function, the trigonometric functions, and the inverse functions of both.

$$y = \tan x \qquad y = bx \cdot \cos x$$

# Successive approximations




**Newton-Raphson method.**

**False Position Method**

Bisection Method

# Solutions to TE

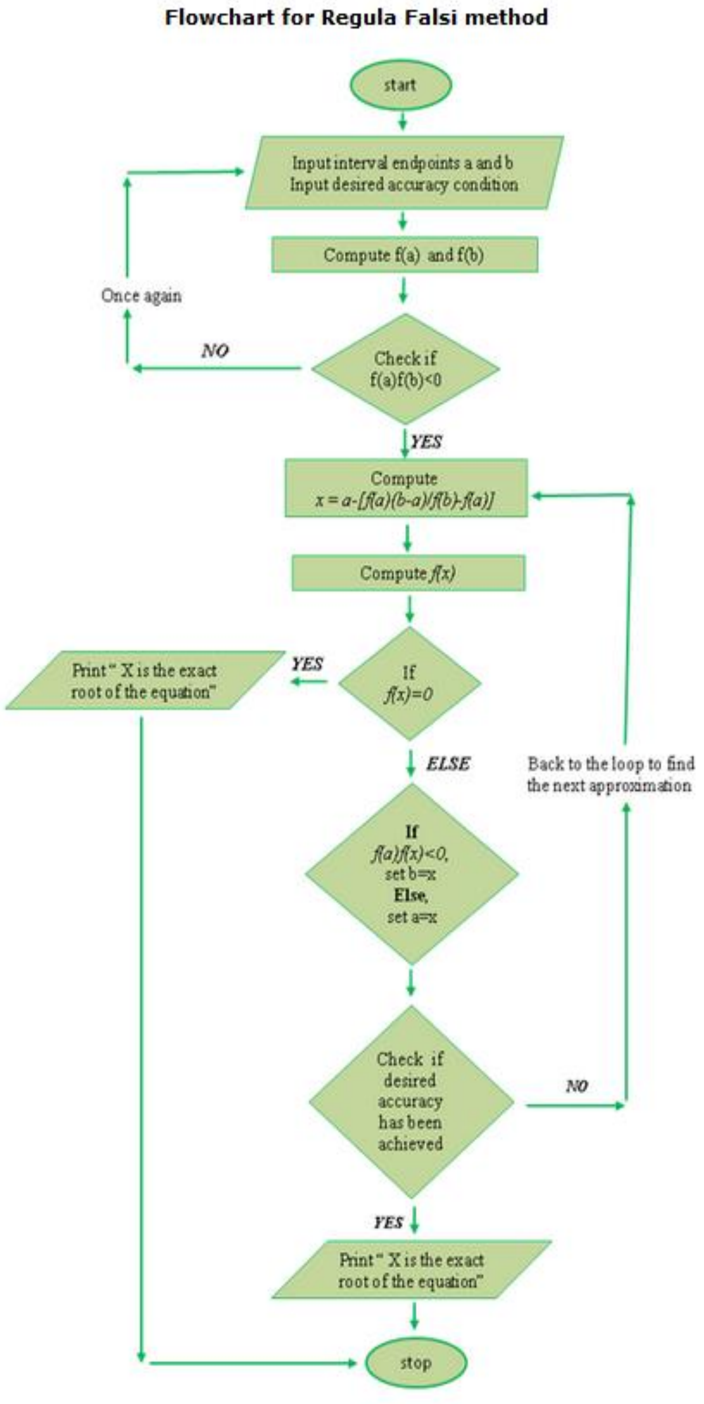
- Newton Raphson method

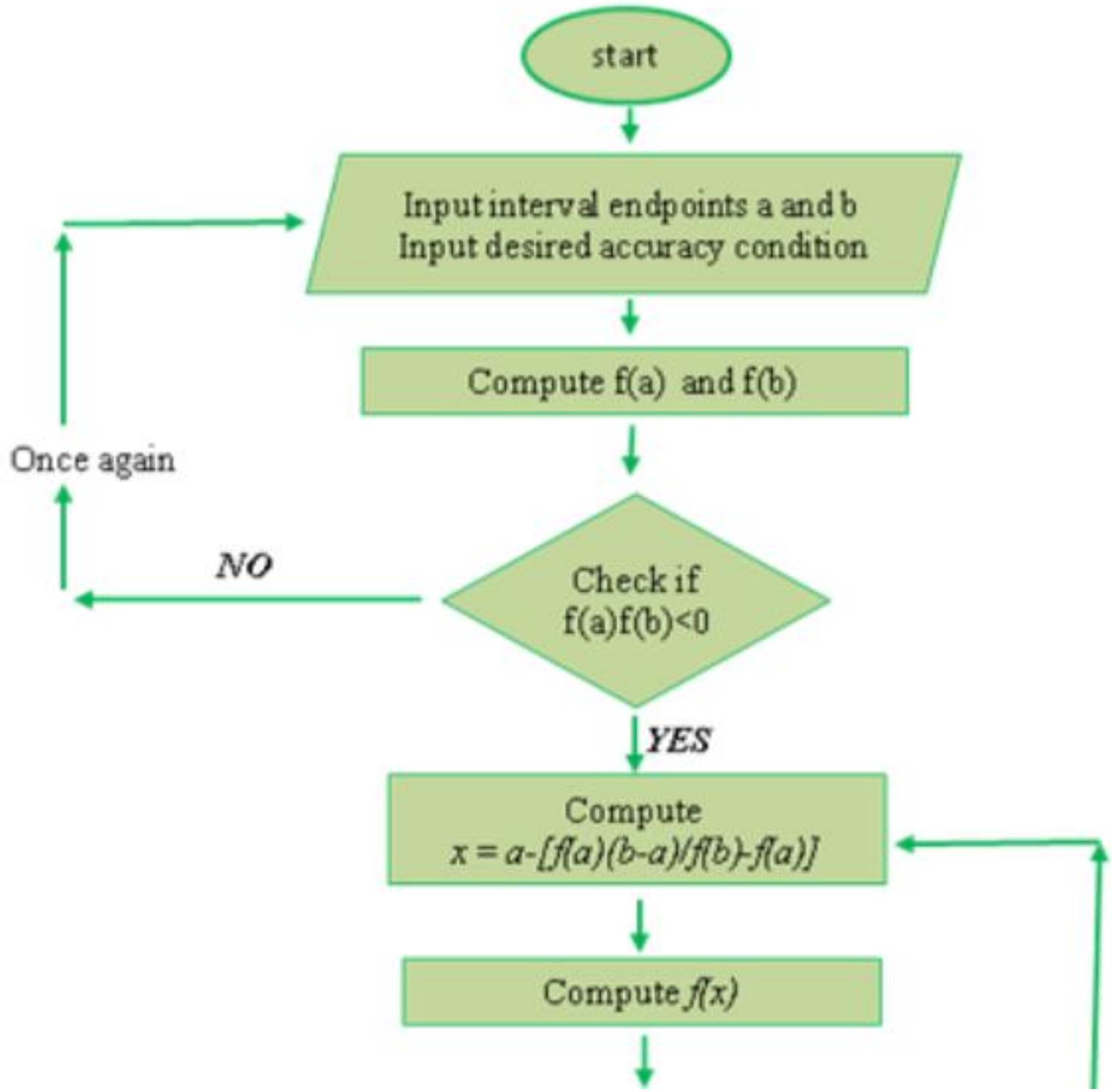
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

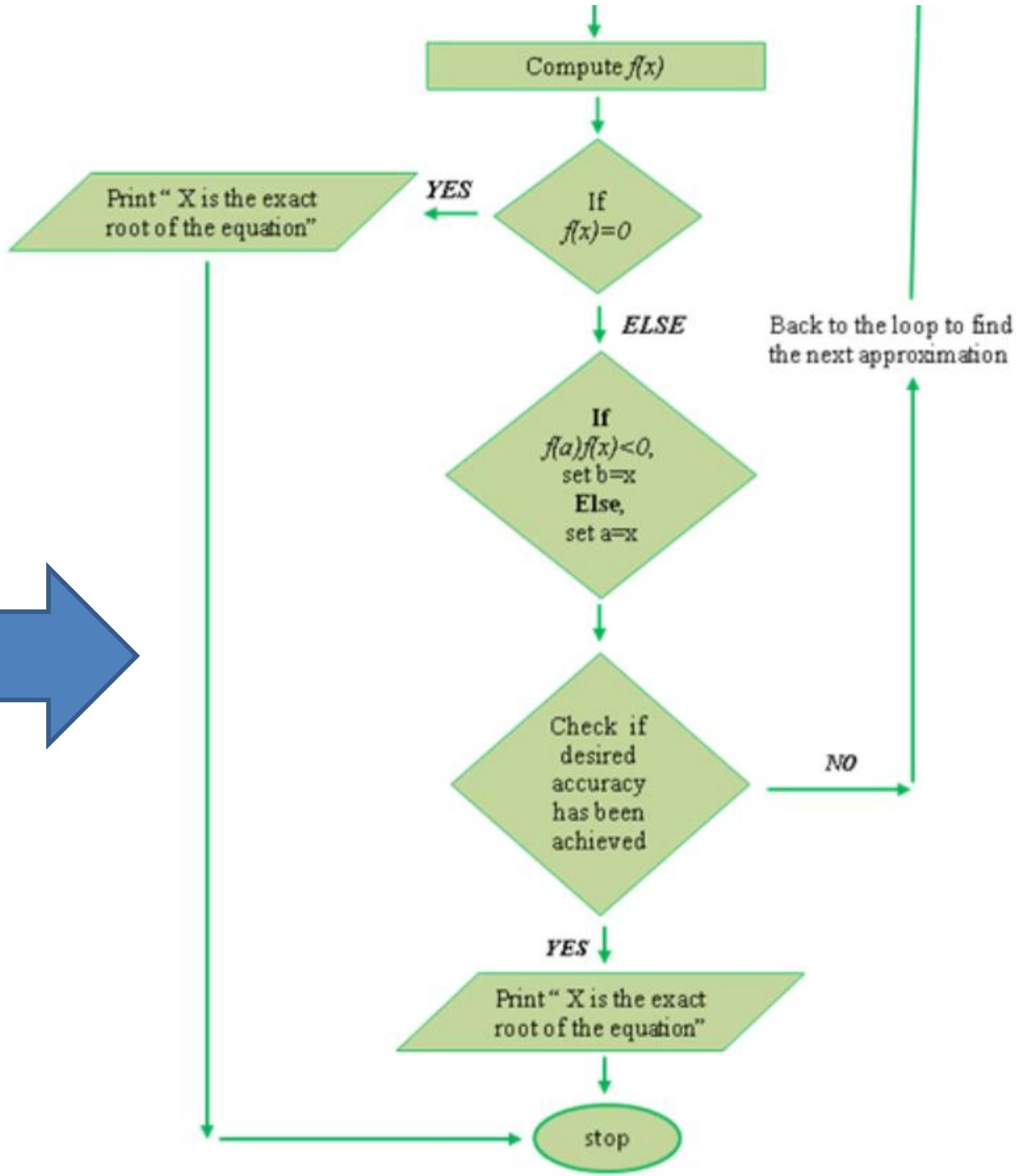
- Regula falsi method

$$x_2 = a_1 - \frac{f(a_1)}{f(b_1) - f(a_1)}(b_1 - a_1)$$

# Regula falsi method

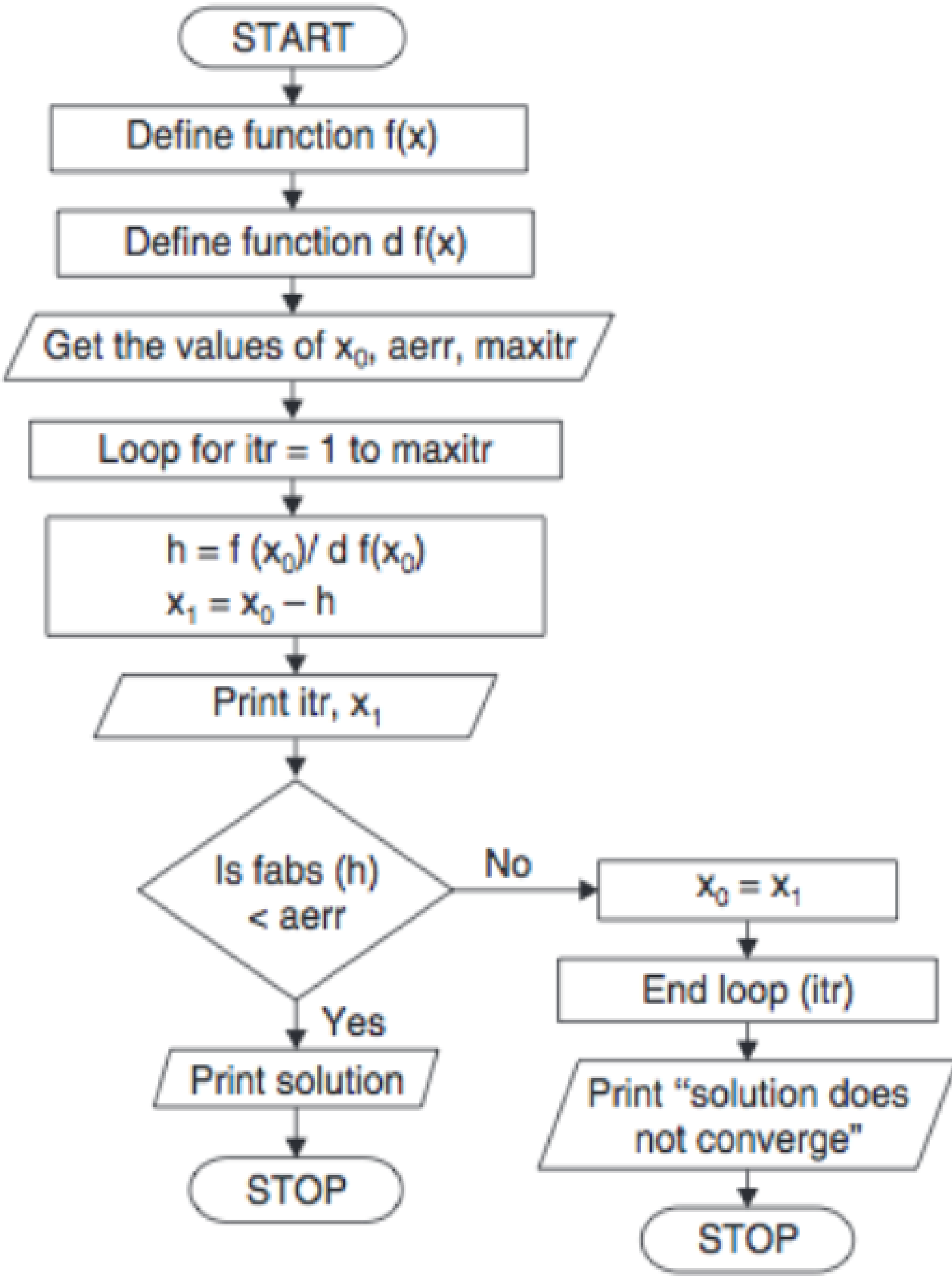



start
Input interval endpoints a and b
Input desired accuracy condition
Compute f(a) and f(b)
Once again
NO
Check if
f(a)f(b)<0
YES
Compute
x = a-[f(a)(b-a)/f(b)-f(a)]
Compute f(x)

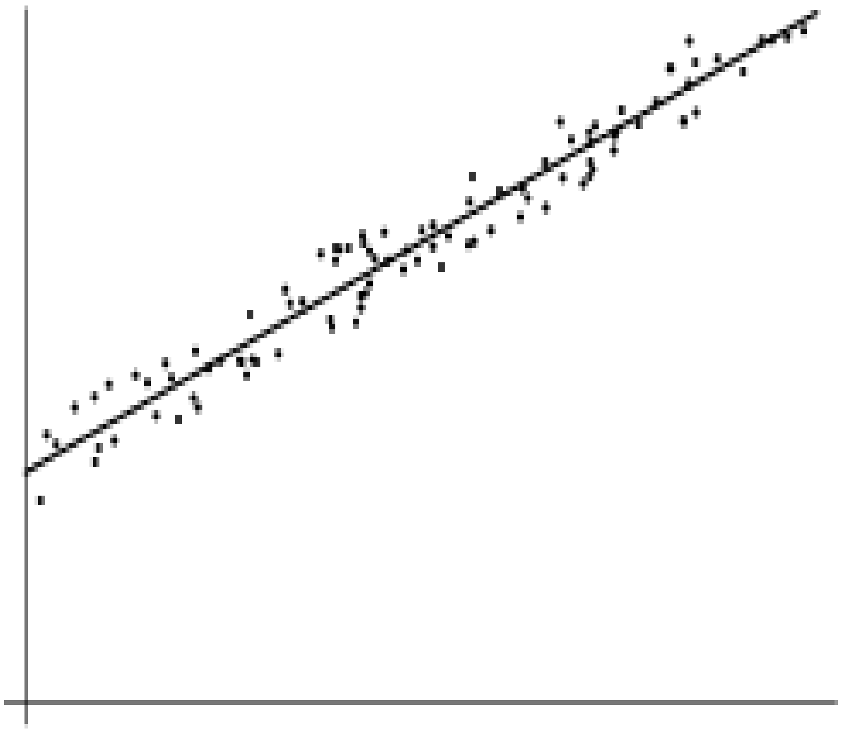


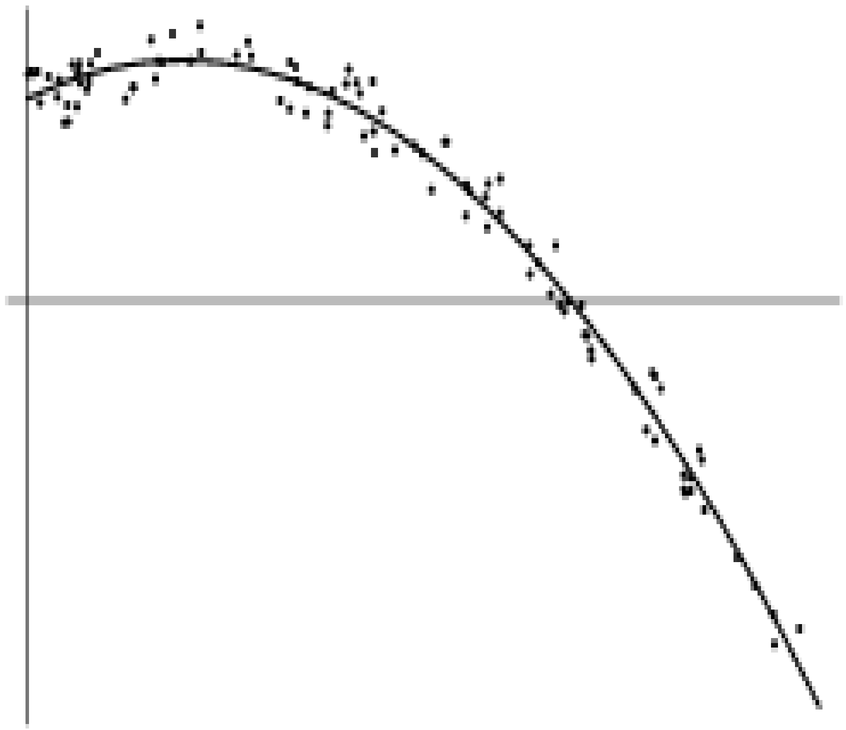
Compute f(x)
If f(x)=0
YES
Print " X is the exact root of the equation"
ELSE
If f(a)f(x)<0, set b=x Else, set a=x
Check if desired accuracy has been achieved
NO
Back to the loop to find the next approximation
YES
Print " X is the exact root of the equation"
stop

# Newton Raphson Method

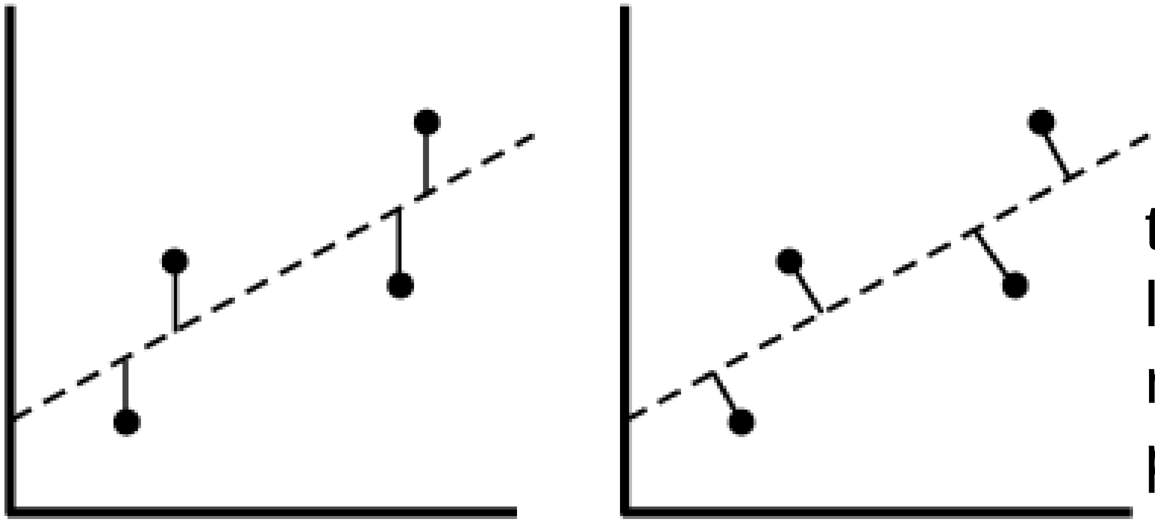


# CURVE FITTING





Best-fitting curve to a given set of points by minimizing the sum of the squares of the offsets of the points from the curve



*vertical offsets* *perpendicular offsets*

the *vertical* offsets from a line are almost always minimized instead of the perpendicular offsets.

The best fit in the least-squares sense minimizes the sum of squared residuals, a residual being the difference between an observed value and the fitted value provided by a model.

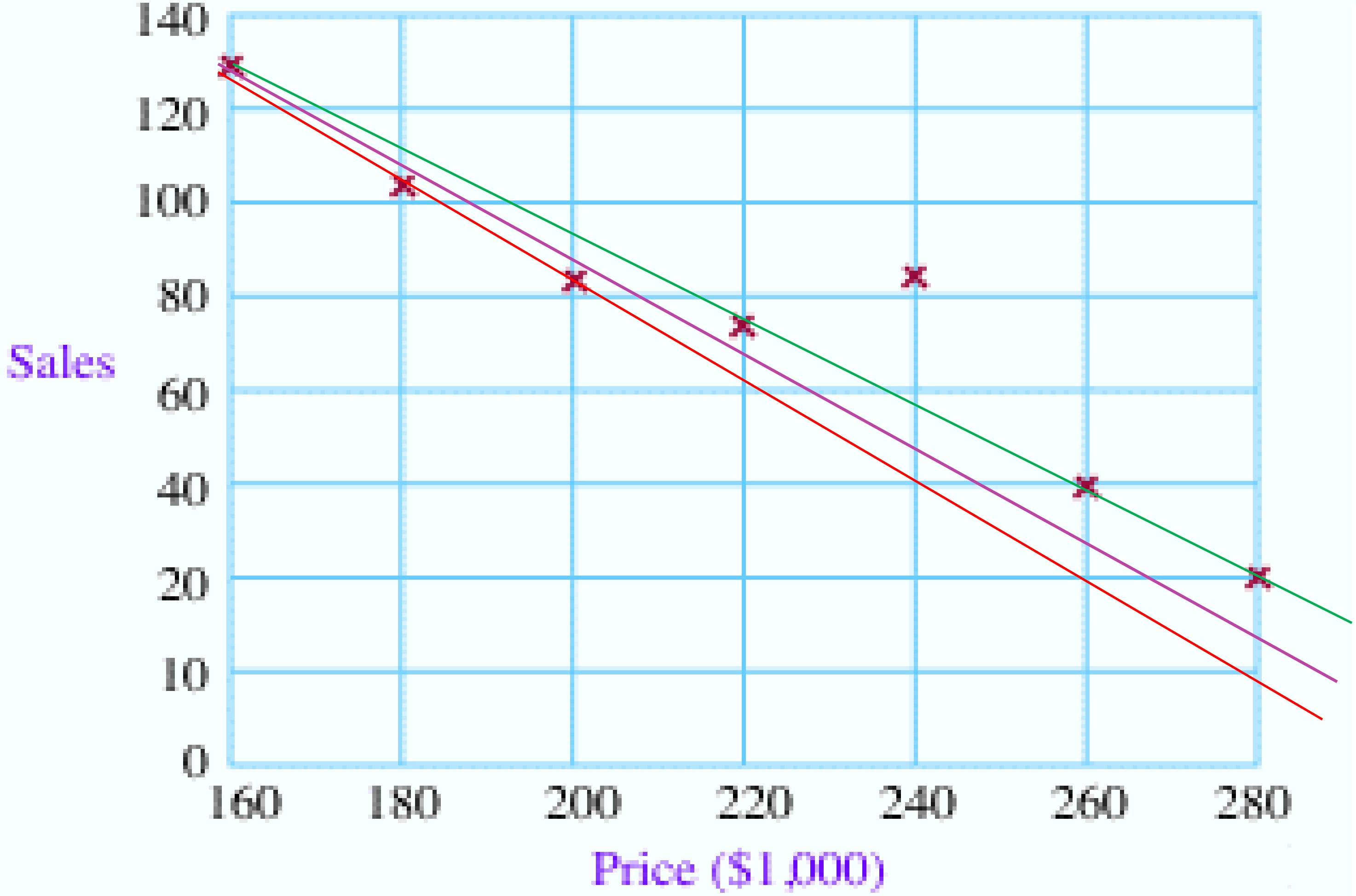
# Minimizing the Residual

$$\text{minimize} \sum |r_i| \qquad \text{or} \qquad \text{minimize} \sum r_i^2$$

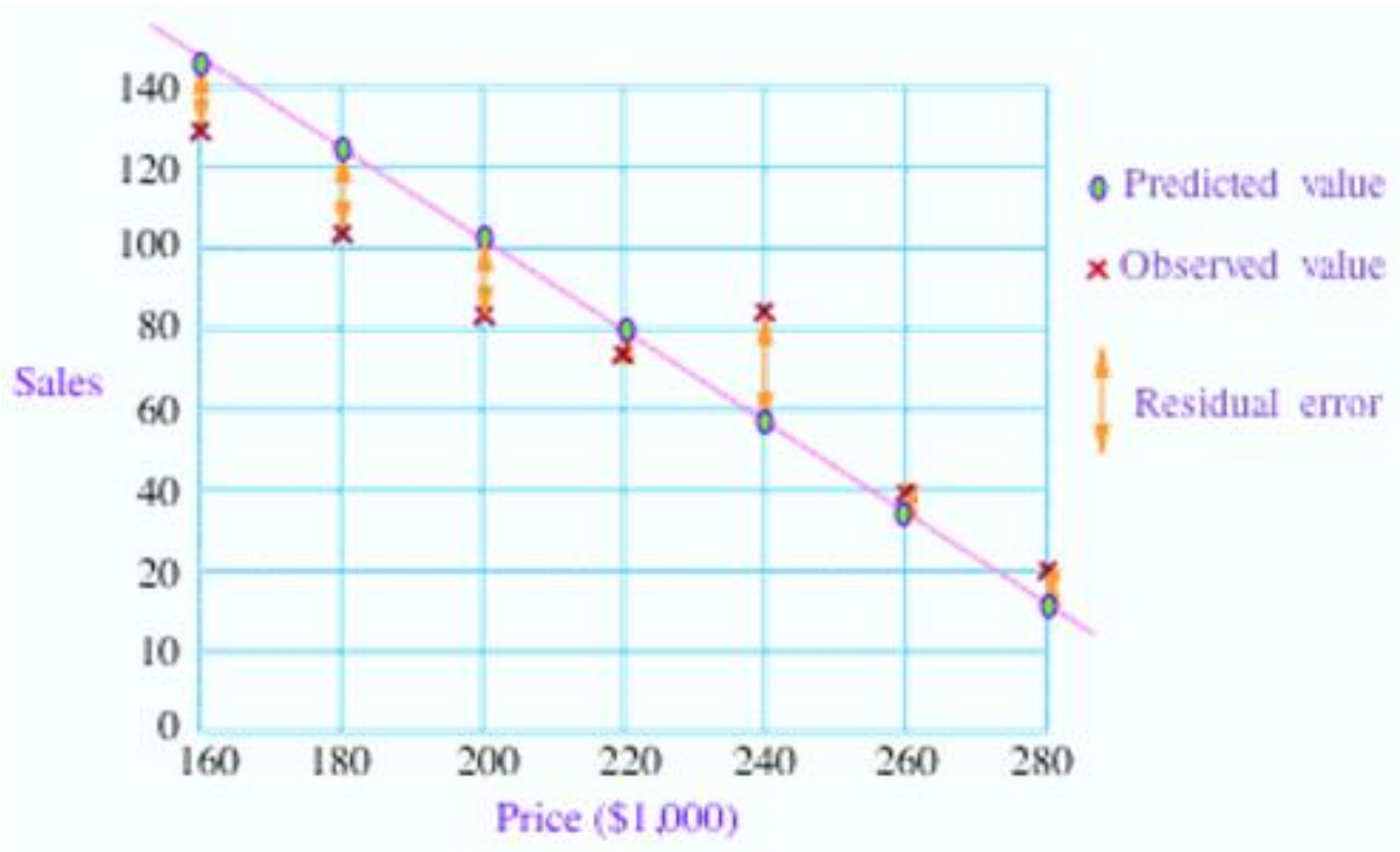
$$\rho = \sum_{i=1}^{m} \left[ y_i - (\alpha x_i + \beta) \right]^2$$

The best fit is obtained by the values of *α and β that minimize ρ.*

| Price (Thousands of $) | 160 | 180 | 200 | 220 | 240 | 260 | 280 |
|---|---|---|---|---|---|---|---|
| Sales of New Homes This Year | 126 | 103 | 82 | 75 | 82 | 40 | 20 |



# Residual Error



# Regression (Best Fit) Line

The best fit line associated with the *n* points ($x1$,$y1$), ($x2$,$y2$) ……………., ($xn$, $yn$) has the form

$y=mx+b$ where

$$\text{Slope} = m = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{n\sum(x^2) - \left(\sum x\right)^2}$$

$$\text{Intercept} = b = \frac{\sum y - m\left(\sum x\right)}{n}$$

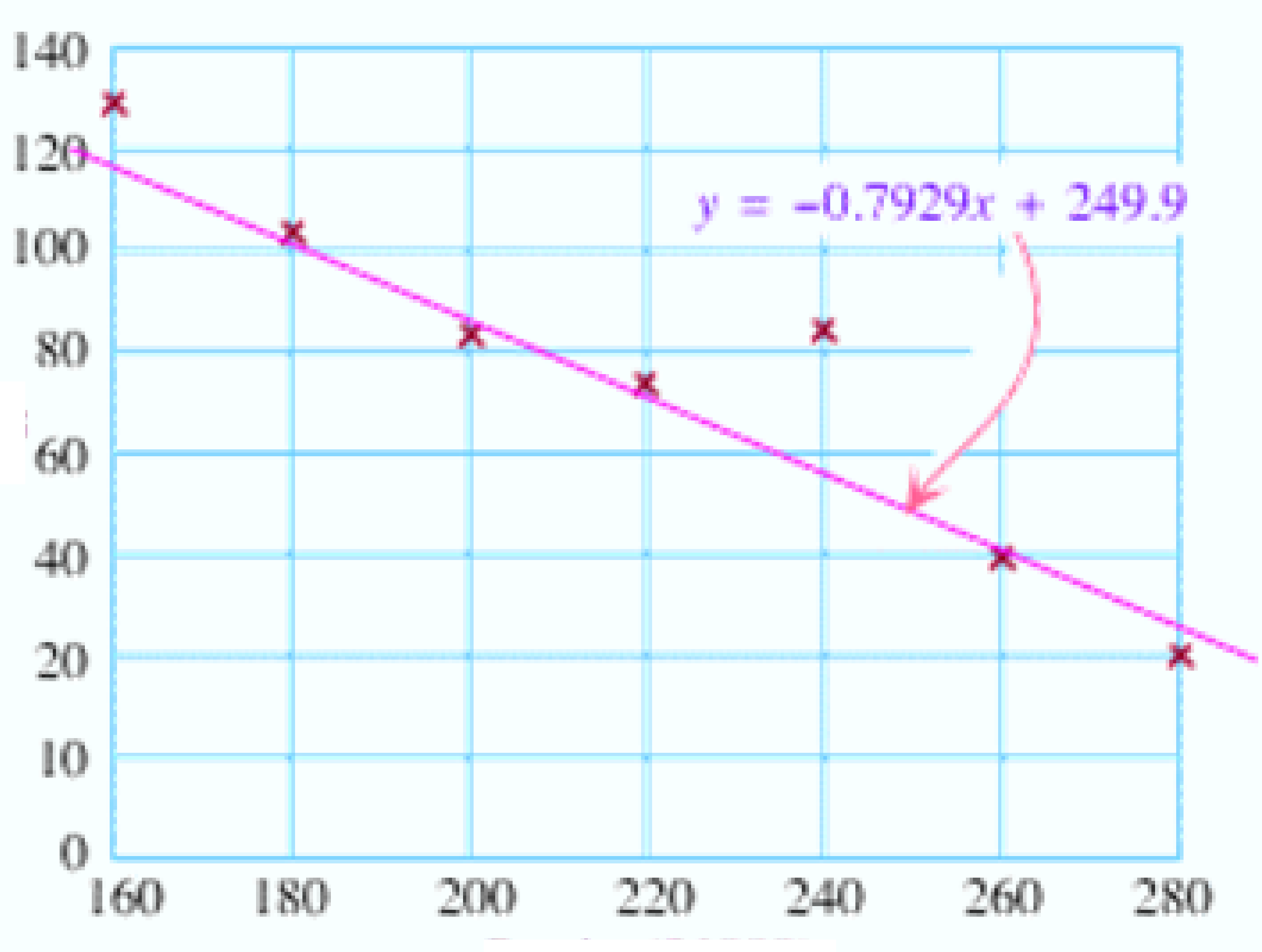
| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| 160 | 126 | 20,160 | 25,600 |
| 180 | 103 | 18,540 | 32,400 |
| 200 | 82 | 16,400 | 40,000 |
| 220 | 75 | 16,500 | 48,400 |
| 240 | 82 | 19,680 | 57,600 |
| 260 | 40 | 10,400 | 67,600 |
| 280 | 20 | 5,600 | 78,400 |
| $\sum x = 1540$ | $\sum y = 528$ | $\sum xy = 107,280$ | $\sum x^2 = 350,000$ |

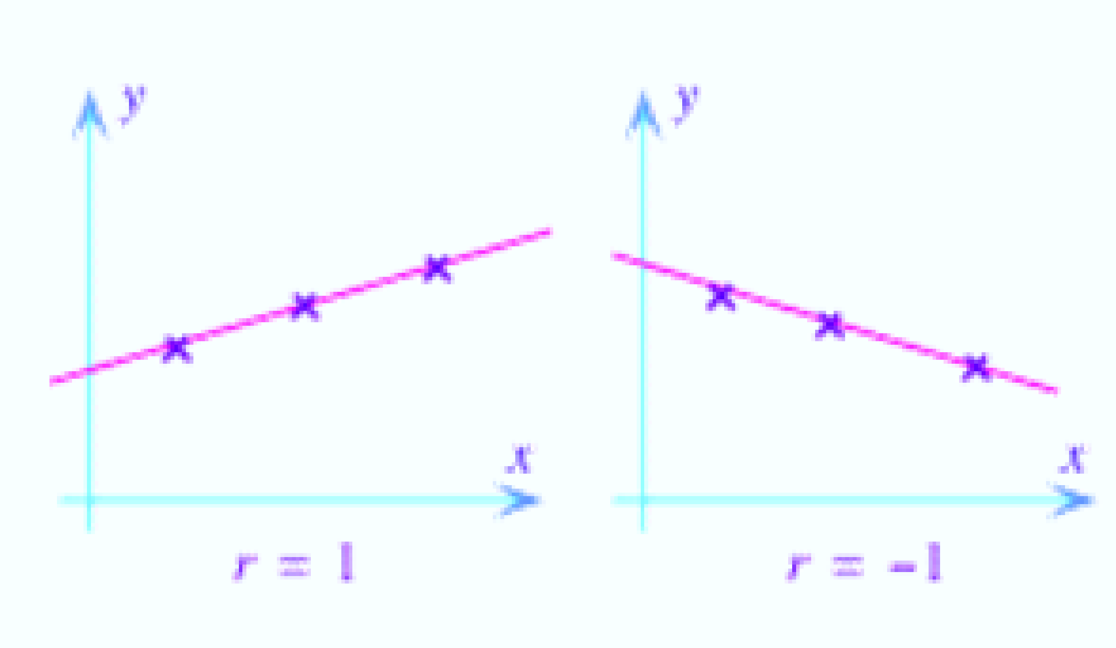
$$m = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{n\sum(x^2) - \left(\sum x\right)^2} = \frac{7(107,280) - (1540)(528)}{7(350,000) - 1540^2} \approx -0.7929$$

$$b = \frac{\sum y - m\left(\sum x\right)}{n} \approx \frac{528 - (-0.7928571429)(1540)}{7} \approx 249.9$$

$$\boxed{y = -0.7929x + 249.9}$$

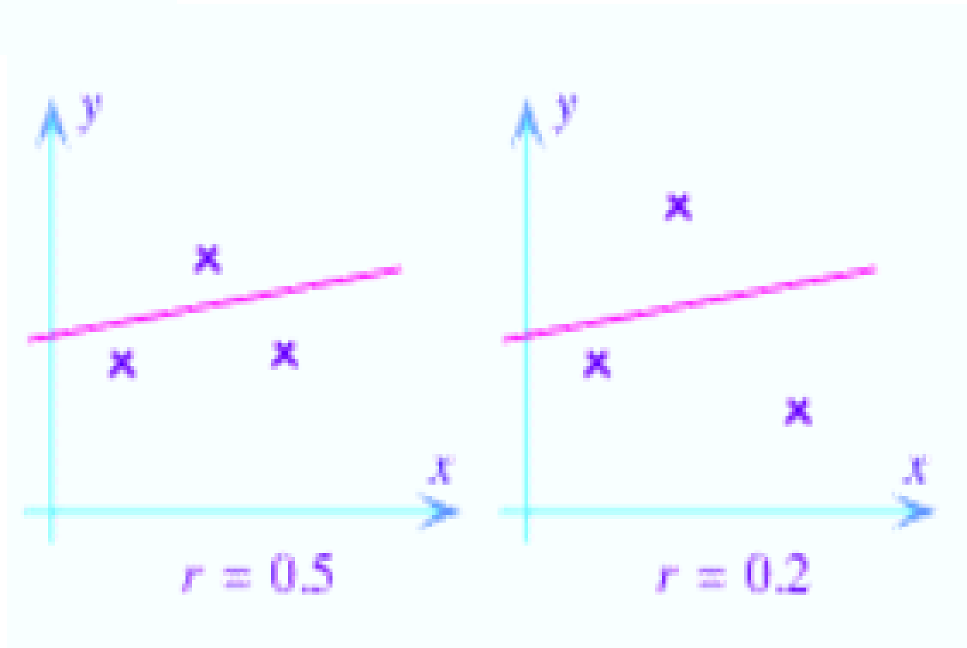
# Least squares line






# Coefficient of correlation

## Goodness of fit



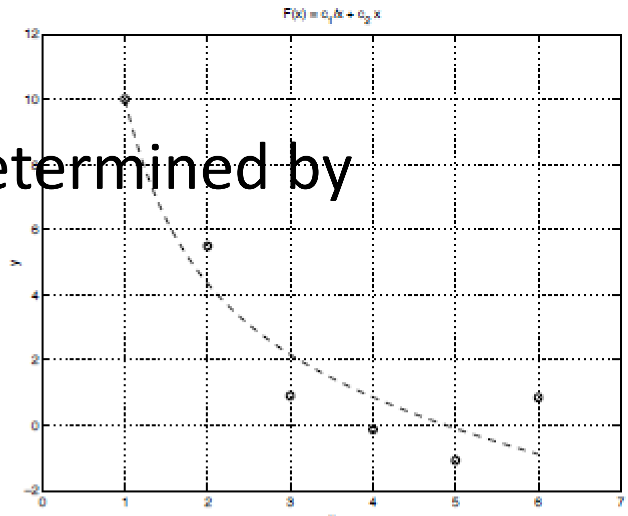

http://www.zweigmedia.com/RealWorld/calctopic1/regression.html

# The least square Parabola

**The least square parabola approximating the set of points $(X_1,Y_1)\ldots(X_n,Y_n)$ has the equation:**

**$Y = a_0 + a_1X + a_2X^2$**

where the constants a0, a1 and a2 are determined by solving simultaneously the equations:



$$\Sigma Y = a_0N + a_1\Sigma X + a_2\Sigma X\hat{}2$$

$$\Sigma XY = a_0\Sigma X + a_1\Sigma X\hat{}2 + a_2\Sigma X\hat{}3$$

$$\Sigma X\hat{}2Y = a_0\Sigma X\hat{}2 + a_1\Sigma X\hat{}3 + a_2\Sigma X\hat{}4$$

Straight Line

Parabola

# Gaussian Elimination

# Gaussian Elimination

- Solving simultaneous linear equations
- solve a general set of *n equations and n* unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
\cdot \qquad\qquad\qquad \cdot \\
\cdot \qquad\qquad\qquad \cdot \\
\cdot \qquad\qquad\qquad \cdot \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

# Steps to Solve

Gaussian elimination consists of two steps

- 1. Forward Elimination of Unknowns: In this step, the unknown is eliminated in each equation starting with the first equation. This way, the equations are *reduced to one* equation and one unknown in each equation.
- 2. Back Substitution: In this step, starting from the last equation, each of the unknowns is found.

# Gaussian Elimination

- 1. Triangulation
  - Upper triangular matrix
- 2. Back Substitution

# INTEGRALS

- Trapezoidal Rule

# INTEGRALS

- Simpson's 1/3rd Rule

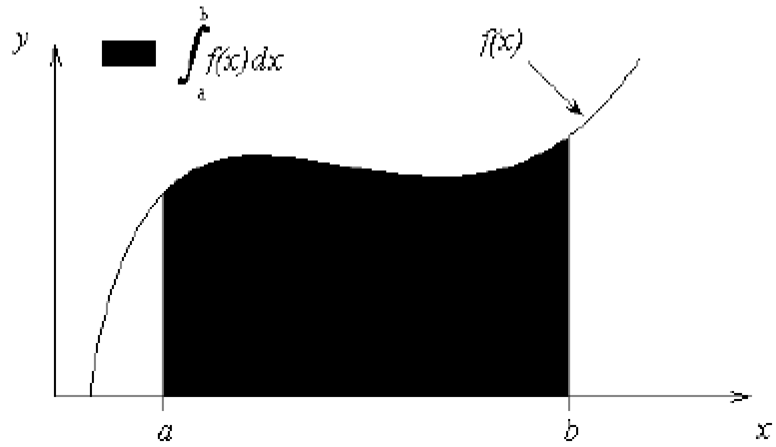
# Gaussian Quadrature

# Integration

- Integration is the process of measuring the area under a function plotted on a graph.
- finding the velocity of a body from acceleration functions,
- displacement of a body from velocity data

$$\int_a^b f(x)dx \cong c_1 f(a) + c_2 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$




http://numericalmethods.eng.usf.edu

# Basis of the Gaussian Quadrature Rule

- The two-point Gauss Quadrature Rule is an extension of the Trapezoidal Rule approximation where the arguments of the function are not predetermined as a and b but as unknowns $x_1$ and $x_2$. In the two-point Gauss Quadrature Rule, the integral is approximated as

$$I = \int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

The four unknowns $x_1$, $x_2$, $c_1$ and $c_2$ are found by assuming that the formula gives exact results for integrating a general third order polynomial,

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

# Basis of the Gaussian Quadrature Rule

It follows that

$$\int_a^b f(x)dx = c_1\left(a_0 + a_1x_1 + a_2x_1^{\;2} + a_3x_1^{\;3}\right) + c_2\left(a_0 + a_1x_2 + a_2x_2^{\;2} + a_3x_2^{\;3}\right)$$

Equating Equations the two previous two expressions yield

$$a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

$$= c_1\left(a_0 + a_1x_1 + a_2x_1^{\;2} + a_3x_1^{\;3}\right) + c_2\left(a_0 + a_1x_2 + a_2x_2^{\;2} + a_3x_2^{\;3}\right)$$

$$= a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) + a_2\left(c_1x_1^{\;2} + c_2x_2^{\;2}\right) + a_3\left(c_1x_1^{\;3} + c_2x_2^{\;3}\right)$$

# Basis of the Gaussian Quadrature Rule

Since the constants $a_0$, $a_1$, $a_2$, $a_3$ are arbitrary

$$b - a = c_1 + c_2$$

$$\frac{b^2 - a^2}{2} = c_1 x_1 + c_2 x_2$$

$$\frac{b^3 - a^3}{3} = c_1 x_1^{\,2} + c_2 x_2^{\,2}$$

$$\frac{b^4 - a^4}{4} = c_1 x_1^{\,3} + c_2 x_2^{\,3}$$

$$x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2}$$

# Gauss Quadrature

**Two-point Gaussian Quadrature Rule**

$$\int_a^b f(x)dx \approx \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right)+\frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right)$$

# Higher Point Gaussian Quadrature Formulas

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

is called the three-point Gauss Quadrature Rule.

The coefficients $c_1$, $c_2$, and $c_3$, and the functional arguments $x_1$, $x_2$, and $x_3$ are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

$$\int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5\right)dx$$

General n-point rules would approximate the integral

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots\ldots + c_n f(x_n)$$

# Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

In handbooks, coefficients and arguments given for n-point Gauss Quadrature Rule are given for integrals

$$\int_{-1}^{1} g(x)dx \cong \sum_{i=1}^{n} c_i g(x_i)$$

as shown in Table 1.

**Table 1: Weighting factors c and function arguments x used in Gauss Quadrature Formulas.**

| Points | Weighting Factors | Function Arguments |
|---|---|---|
| 2 | $c_1$ = 1.000000000<br>$c_2$ = 1.000000000 | $x_1$ = -0.577350269<br>$x_2$ = 0.577350269 |
| 3 | $c_1$ = 0.555555556<br>$c_2$ = 0.888888889<br>$c_3$ = 0.555555556 | $x_1$ = -0.774596669<br>$x_2$ = 0.000000000<br>$x_3$ = 0.774596669 |
| 4 | $c_1$ = 0.347854845<br>$c_2$ = 0.652145155<br>$c_3$ = 0.652145155<br>$c_4$ = 0.347854845 | $x_1$ = -0.861136312<br>$x_2$ = -0.339981044<br>$x_3$ = 0.339981044<br>$x_4$ = 0.861136312 |

# Arguments and Weighing Factors

The table is given for $\int_{-1}^{1} g(x)dx$ integrals, how does one solve $\int_{a}^{b} f(x)dx$ ?

The answer lies in that any integral with limits of $[a,b]$

can be converted into an integral with limits $[-1,1]$

Let $x = mt + c$

If $x = a,$ then $t = -1$

If $x = b,$ then $t = 1$

Such that: $m = \frac{b-a}{2}$

Then $c = \frac{b+a}{2}$ Hence $x = \frac{b-a}{2}t + \frac{b+a}{2}$ $dx = \frac{b-a}{2}dt$

Substituting our values of x, and dx into the integral gives us

$$\int_{a}^{b} f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)\frac{b-a}{2}dt$$

Gaussian Quadrature